Dynamic Stability of Circular Plates under Stochastic Excitations

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This paper deals with a Lyapunov-type analysis of a linear structural dynamic system which is excited by a stochastic parametric load. The technique of solution is developed on the basis of the linearity of the elastic system but with modification in developing the Lyapunov function, the technique can be used to consider nonlinear systems. Stochastic convergence and a theory for mean square global stability are discussed. A radially loaded circular plate is considered, whose dynamic motion can be described by a Hill-type equation, with one parameter being a random variable. A Lyapunov function, satisfying the stochastic stability theorem, is developed and sufficient conditions are established to insure mean square global stability. These conditions are applicable for any general continuous random process, and are expressed in terms of the excitation, physical parameters of the system and a measure of the stochastic dependency of the output on the input.

Introduction

THE dynamic stability of plates subjected to deterministic loads has been considered rather thoroughly by Bolotin[1] and also provides an excellent bibliography on the subject. The question of stability of plates when the loads are not periodic but statistical in nature has not been investigated to date. Although, in recent years, much effort has been devoted to the statistical properties of linear and nonlinear systems subjected to stochastic loads.[2–4] The results of such analysis are valid only where the parameters characterizing the system are deterministic. There are few discussions available of the case where the system parameters, themselves, are random functions of time. The work described in this paper is an extension of work done by the authors, reported in August 1968, in an earlier paper,[5] in which the stability of an axially loaded column was investigated. In this paper we use classical methods to present the small linear deformations of a thin circular plate subjected to radial, randomly timed varying loads. We consider the stochastic stability of a circular plate by using a Lyapunov type approach, an extension of the work of Bertram and Sarachik.[6] We reduce the classical differential equation of damped linear plate theory to Hill-type equation. This is accomplished by using Galerkin's variational techniques. Basically the problem is to determine the general behavior of a system of differential equations as time increases. In particular, we are concerned with establishing sufficient conditions guaranteeing stochastic stability—stability in mean square—of a circular plate subjected to radial, stochastic excitation. This problem has not appeared in the literature before.

Stochastic Stability

As with systems of equations governed by differential equations with deterministic coefficients, the concept of stability here is still essentially a matter of convergence of solutions of the system, but deals with limits involving random variables. The sequence of random variables x_n converges to x in the mean square if $E[\bar{x} - x] \to 0$ as $n \to 0$. Thus, we are primarily concerned with stability which corresponds to convergence in mean square of the system described by $\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x},q(t),t)]$ where $\mathbf{x}$ is a vector describing the state of the system, $\mathbf{f}(\mathbf{x},q,t)$ is a continuous vector function of the stochastic variable $q(t_0)$. $\mathbf{f}(\mathbf{x},q,t)$ satisfies a Lipshultz condition and such that $\mathbf{f}[0,q(t),t] = 0$ for all t. The null solution $\mathbf{x}(t) = 0$ is the equilibrium solution to be investigated.

Hence, we can write the following definition: an equilibrium state $\mathbf{x}(t) = 0$ is globally stable in mean square if for every real $\epsilon > 0$, there exists a real number $\delta(\epsilon) > 0$ such that for any finite initial state, whose norm satisfies $||\mathbf{x}^2(t_0)|| < \delta(\epsilon)$ the relationship:

$$E[x^2(t)] < \epsilon \text{ and } \lim_{t\to\infty} E[||x^2(t)||] \to 0$$

Using this definition of stability, the behavior of a stochastic system can be determined if a Lyapunov function can be found which satisfies the theorem.‡ If there exists a Lyapunov function $V(x,t)$ defined over the entire state space, with the following properties: 1) $V(0,t) = 0$, 2) $V(\mathbf{x},t)$ is continuous in mean square in both variables $\mathbf{x}$ and t, and the first partial derivative of $V(x,t)$ in its variables exists, 3) $V(\mathbf{x},t)$ is a positive definite function, i.e., $V(\mathbf{x},t) \geq a\, x^2(t)$ for some a, and 4) $\lim V(\mathbf{x},t) \to \infty$ for $||x(t)|| \to \infty$. Then the equilibrium solution $\mathbf{x}(t) = 0$ is globally stable in the mean square sense if $V(\mathbf{x},t)$ is a decresent function whose expected value of its derivative dV/dt is negative definite over the entire state space of $\mathbf{x}(t)$, i.e., $E[\dot{V}(\mathbf{x},t)] < 0$ for all $||\mathbf{x}(t_0)||\, t$.

Stability Analysis of Circular Plates

For a thin circular plate subjected to a stochastic radial load $R(t)$, we assume that the stochastic process $[R(t): t\epsilon(0,\infty)]$ is continuous in the interval 0 to ∞ with probability 1 and that it is strictly stationary. We further make those assumptions

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‡ Proofs of this theorem are given, essentially, by Bertram and Sarachik.[6] While they speak about stability in the mean sense, we speak of stability in the mean square sense.

customarily made in the deterministic study of classical plate theory.

With these assumptions the governing differential equation of classical damped linear plate theory is well known to be

$$D\nabla^4 u + m\partial^2 u/\partial t^2 + c\partial u/\partial t = q + F \quad (1)$$

where $u(r,\psi,t)$ = displacement of the middle plane, m = mass of the plate per unit area, h = thickness of the plate, c = viscous damping coefficient, F = body forces acting on the plate, and q is a fictitious load, taking into consideration the effect of the radial load

$$\therefore q = N\partial^2 u/\partial r^2 + N_\psi[(1/r^2)\partial^2 u/\partial\psi^2 + (1/r)\partial u/\partial r] \quad (2)$$

where $N_r = N_\psi = -R(t)$. The body forces, exclusive of the inertia and damping effects, are assumed to be zero. Also, there is no transverse loading assumed. Equation (1) can now be written

$$\nabla^2[D\nabla^2 + R(t)]u = -m(\partial^2 u/\partial t^2) - c(\partial u/\partial t) \quad (3)$$

Considering the free transverse vibration of the plate, we can write Eq. (3) as

$$\nabla^4 u + (m/D)(\partial^2 u/\partial t^2) = 0 \quad (4)$$

Assuming the solution to Eq. (4) to be of the form

$$u(r,\psi,t) = V(r,\psi)e^{iwt} \quad (5)$$

We have the following:

$$(\nabla^2 + r^2)(\nabla^2 - r^2)V = 0 \quad (6)$$

where $r^4 = \omega^2 m/D$. Therefore, U can be a solution of either $\nabla^2 U + r^2 = 0$ or of $\nabla^2 U - r^2 = 0$. We can write

$$\partial^2 U/\partial r^2 + (1/r)(\partial U/\partial r) + (1/r^2)(\partial^2 U/\partial\psi^2) \pm r^2 U = 0 \quad (7)$$

For our application, we specify a completely circular plate, subjected to the following boundary conditions:

$$U(R,\psi) = 0 \quad (8)$$

$$(\partial U/\partial r)(R,\psi) = 0 \quad (9)$$

where R is the radius of the plate. The possible solutions of Eq. (7) can be assumed in the following form:

$$U_n(r,\psi) = \varphi_n(r)\cos n\psi \quad (10)$$

which form a complete set, satisfying the boundary conditions Eqs. (8) and (9). Equation (7) then becomes

$$\varphi''_n + (1/r)\varphi'_n + (\pm r^2 + n^2/r^2)\varphi_n = 0 \quad (11)$$

If we impose the condition that we have an axially symmetric loading on the plate, our solution to Eq. (11) can be written as

$$\varphi_m(r) = A_m J_0(r_m k) + C_m I_0(r_m r) \quad (12)$$

where J_m and I_0 are Bessel functions for real and imaginary arguments. Equation (12) must satisfy the conditions

$$\varphi_m(R) = 0 \quad (13)$$

$$\partial(\varphi m/\partial r)(R) = 0 \quad (14)$$

This yields the solution Eq. (12) to be

$$\varphi_m(r) = I_0(r_m R)J_0(r_m r) - J_0(r_m R)I_0(r_m r) \quad (15)$$

with the following eigenvalues: $\Gamma_1 = r_1 R = 3.1871$, $\Gamma_2 = r_2 R = 6.3020$, $\Gamma_3 = r_3 R = 9.4236$,

$$\Gamma m = rmRm \rightarrow m\pi \quad (16)$$

The natural frequencies are determined by

$$\omega m = (D/m)^{1/2} r_m^2 \quad (17)$$

We seek a solution to Eq. (3) which is separable in the time variable and in a free variation mode, i.e.,

$$u(r,t) = f(t)\varphi(r) \quad (18)$$

where $\varphi(r)$ is defined by Eq. (15). Equation (18) will not satisfy Eq. (3), since the free vibration forms and the buckling modes are not the same. In accordance with the Galerkin variational method,[7] we substitute Eq. (18) into Eq. (3) and require the resulting equation to be orthogonal to the free vibration characteristic function $\varphi(r)$. To this end, we have, upon letting $\tau = wt$,

$$f''(t) + 2\Omega pf'(t) + [1 - 2\beta_p R(t)]f(t) = 0 \quad (19)$$

where

$$\Omega_p = C/2mw,\ \beta_p = \tfrac{1}{2}R_0 \quad (20)$$

$$R_0 = -m\omega^2\int_0^R r\varphi^2(r)dr \Big/ \int_0^R r\varphi(r)\nabla^2\varphi(r)dr \quad (21)$$

and the value for R_0, for any mode, is determined to be

$$R_C = \frac{\Gamma}{R^2}D\left[\frac{I_0^2(\Gamma)J_0^2(\Gamma) + I_0^2(\Gamma)J_1^2(\Gamma) - J_0^2(\Gamma)I_1^2(\Gamma)}{I_0^2(\Gamma)J_1^2(\Gamma) + I_1^2(\Gamma)J_0^2(\Gamma)}\right] \quad (22)$$

It can be shown that the minimum value of R_0 occurs when $\Gamma = rR$ is minimum. Therefore, when $\Gamma = 3.1871$

$$R_0 = 14.81D/R^2 \quad (23)$$

From Ref. 8, the static critical load is

$$R_{cr} \approx 14.78D/R^2 \quad (24)$$

or 0.9% less. We can then write Eq. (19) as

$$f''(t) + 2\Omega_p f'(t) + [1 - 2\beta_s R(t)]f(t) = 0 \quad (25)$$

where $\beta_s = \frac{1}{2}R_{cr}$.

The question of stability of the radially loaded plate is now reduced to that of investigating the behavior of Eq. (25) by Lyapunov's methods, as previously described, to establish stability criteria. For convenience, we drop the subscripts from Eq. (24), and we can reduce it to

$$\frac{d}{d\xi}\begin{Bmatrix} z_1 \\ {}_2 \end{Bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2\Omega \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + 2\beta R(t)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad (26)$$

$$\therefore \mathbf{Z}' = [\tilde{A} + F(t)\tilde{B}]\mathbf{z} \quad (27)$$

$$\therefore \mathbf{Z}' = \tilde{D}(t)\mathbf{Z} \quad (28)$$

In developing the Lyapunov function $V(z,t)$ for the system described by Eq. (28), we select as a possible function the quadratic form which is defined over the entire state space

$$V(\mathbf{z},t) = \mathbf{z}^T\tilde{Y}\mathbf{Z} \quad (29)$$

where $\tilde{Y}$ is a real symmetric positive definite matrix and $\mathbf{z}$ and $\mathbf{z}^T$ are the state vector and its transform, respectively. In this form, the Lyapunov function approaches zero as the state vector approaches zero, for all time, i.e., descrescent. Hence, all the conditions of our theorem are satisfied except that of the behavior of the mean value of the time derivative, which is what we now consider.

The time derivative of Eq. (29) is written as

$$V(\mathbf{z},t) = \omega[\mathbf{z}^T\tilde{Y}\mathbf{z} + \mathbf{z}^T\tilde{Y}\mathbf{z}'] \quad (30)$$

evaluating this along the trajectory Eq. (28), we obtain

$$V(z,t) = \omega\{\mathbf{z}^T[\tilde{A}^T\tilde{Y} + \tilde{Y}\tilde{A}]\mathbf{z} + \mathbf{z}^T[\tilde{B}^T\tilde{Y} + \tilde{Y}\tilde{B}]\mathbf{z}F(t)\} \quad (31)$$

We assume that Eq. (29) is the Lyapunov function satisfying the stability requirements for the constant coefficient por-

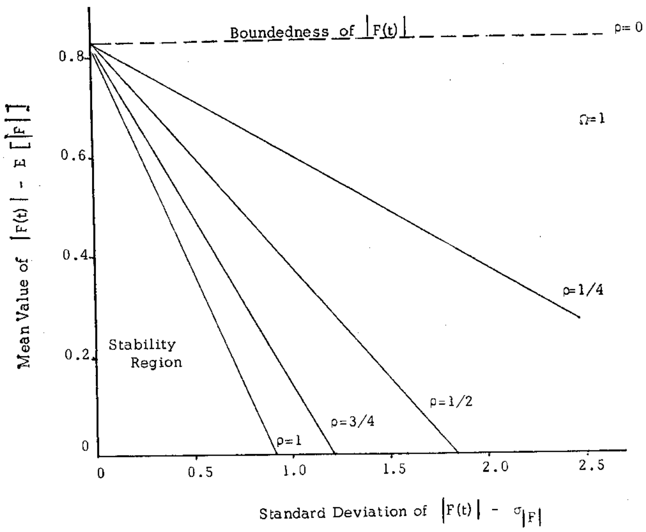

Fig. 1 Regions of mean square for general probability distribution.

tion of the system described by Eq. (28)

$$\mathbf{z}' = \tilde{A}\mathbf{z} \tag{32}$$

The negative definiteness of V will be assured if the matrix sum in the brackets is negative definite. Hence, we have

$$\tilde{A}^T\tilde{Y} + \tilde{Y}\tilde{A} = -\tilde{C} \tag{33}$$

where C is a positive definite matrix and for this case is assumed to be the unit matrix I. It has been shown by Alimov[9] that the V function determined in the aforementioned manner is admissible as a Lyapunov function. The three unknown elements of the matrix Y can now be determined by the linear equations of Eq. (33), with $C = I$:

$$Y = \begin{bmatrix} \Omega + \frac{1}{2}\Omega & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\Omega \end{bmatrix} \tag{34}$$

Having our V function completely defined we rewrite Eq. (31),

$$\dot{V} = \omega\{-\mathbf{z}^T\mathbf{z} + \mathbf{z}^T\tilde{H}\mathbf{z}F_{(2)}\} \tag{35}$$

where

$$\tilde{H} = \tilde{B}^T\tilde{Y} + \tilde{Y}\tilde{B} \tag{36}$$

Since $\tilde{Y}$ is a real symmetric positive definite matrix and the diagonal terms of the matrix $\tilde{B}$ are all zero we conclude that $\tilde{H}$ is also a symmetric matrix and hence there exists an orthogonal transformation φ such that the following relationships are valid:

$$\tilde{\varphi}^T\tilde{H}\tilde{\varphi} = \tilde{\lambda};\ \tilde{\varphi}^T\tilde{\varphi} = \tilde{I} \tag{37}$$

where $\tilde{\lambda}$ is a diagonal matrix of the eigenvalues of H. Letting

$$\mathbf{z} = \tilde{\varphi}\eta \tag{38}$$

we obtain

$$V(\mathbf{z},t) = \omega\{\eta^T\eta + \eta^T\tilde{\lambda}\eta F(t)\} \tag{39}$$

which can be written as

$$\dot{V}(z,t) \leq \omega\sum_{i=1}[F(t)\lambda_i -]\eta_i^2 \tag{40}$$

Finally, to completely satisfy our theorem for a global stability in mean square we require the mean value, of the function $\dot{V}(z,t)$ to be negative definite. To this end we obtain

$$E[\dot{V}(z,t)] \leq \omega\sum_{i=1}|\lambda_i|E[|F(t)|\eta_i^2] - E[n_i^2] \tag{41}$$

Hence, for stability in mean square we must satisfy the following criterion which is imposed on the system and which will guarantee stability:

$$|\lambda_{max}|E[|F(t)|n^2] - E[n^2] < 0 \tag{42}$$

where

$$n^2 = \sum_{i=1}\eta_i^2 = \sum_{i=1}z_i^2 \tag{43}$$

since the magnitude of the state vector is invariant to a coordinate transformation.

The condition for global stability in mean square can be expressed as

$$E[|F(t)|n^2] - (1/\lambda_{max})E[n^2] < 0 \tag{44}$$

In this form, statistical bounds on the excitation are not easily achieved. We introduce the correlation coefficient and cross covariance of $F(t)$ and n^2. The condition for global stability becomes

$$E[|F(rt)|] + \zeta[E(n)/E^2(n^2) - 1] + \sigma|F(t)| < 1/\lambda_{max} \tag{45}$$

The maximum eigenvalue of $\tilde{H}$ is

$$\tilde{H} = \begin{bmatrix} 1 & \frac{1}{2}\Omega \\ \frac{1}{2}\Omega & 0 \end{bmatrix} \tag{46}$$

$$\therefore \lambda_{max} = \tfrac{1}{2}[1 + (1 + 1/\Omega^2)\tfrac{1}{2}] \tag{47}$$

Substituting this into Eq. (45), our criterion becomes

$$E[|R(t)|] + \rho[E(n^4)/E^2(n^2) - 1]^{1/2}\sigma|R(t)| < 2R^0/[1 + (1 - 1/\Omega 2)^{1/2}] \tag{48}$$

where R^0 is the static buckling load.

The criterion specified in Eq. (48) is rigorous for any continuous, stationary random process. The difficulty in applying Eq. (48) in the form given is apparent. To determine precisely the value of the coefficient of the standard deviation, $\sigma R(t)$, of the excitation, we must know the distribution function function of the norm of the state vector n. Since, in general this is not possible, we assume a minimum biased distribution, i.e., that the response measure n is uniformly dis-

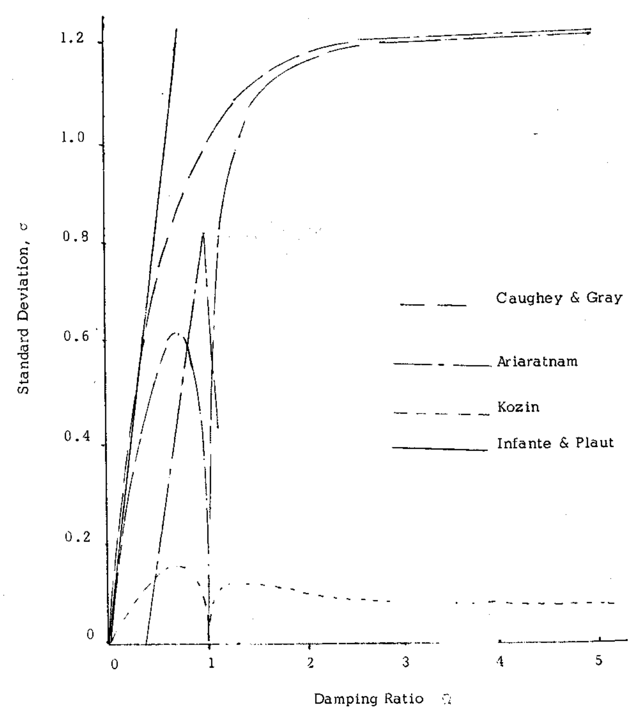

Fig. 2 Almost sure stability bounds for gaussian processes.

tributed in an arbitrary interval $(0,n_0)$. We now are able to determine the required moments of n;

$$E[n^4] = n_0^4/5; \; E[n^2] = n_0^2/3 \quad (49)$$

Therefore, with the above assumption, Eq. (48) reduces to the following norm:

$$E[|R(t)|] + 0.895\rho\sigma_{|R(t)|} < 2R^0/[1 + (1 - 1/\Omega^2)^{1/2}] \quad (50)$$

Since for our case, we would expect positive correlation between the excitation and response we can consider only $\rho \geq 0$.

General Probability Distribution

The foregoing sufficient conditions are applicable for any continuous random excitation. Certain simplifications can be achieved when various individual types of stochastic excitation processes are assumed. But let us investigate the general behavior of condition Eq. (50).

In Fig. 1, the relationship is demonstrated between the mean of the absolute value of the excitation and its standard deviation for a particular value of the damping coefficient $\Omega = m/\omega$ and for various values of the correlation coefficient ρ. The excitation is normalized with respect to the Euler critical load R^0, i.e., $F(t) = R(t)/R^0$. The regions enclosed by a particular ρ curve and the coordinate axes, we shall define as regions of mean square global stability. It is apparent that as the variance of the excitation decreases, the corresponding mean value increases, as we would expect. Moreover the regions of stability increase in scope as the correlation between the two random variables decreases. In the limit, the stability bounds coincide with the boundedness values of the excitation.

There is no basis for comparison of magnitudes of the above results for mean-square stability. To provide a basis for comparison of the results from our conditions with results obtained by others, we consider the case where the mean square stability bounds become

$$(0.798 + 0.540)\sigma_F < 2/[1 + (1 + 1/\Omega^2)]^{1/2} \quad (51)$$

Stability bounds of other authors is shown in Fig. 2. All four authors considered the concept of almost sure global stability. Both Kozin[11] and Ariaratnam[12] apply the Gronwall-Bellman Lemma to the integral solution of Eq. (24) to obtain conditions for almost sure stability. The bounds of Kozin are considerably smaller than the other authors, and he shows an unusual decrease near $\Omega = 1$. These characteristics are a consequence of Kozin's choice of an upper bound on the magnitude of the impulse response function and not characteristic of the system under consideration. Ariaratnam shows that the bounds can be extended and the sharp decrease eliminated by introducing three separate rages of the damping coefficient, Ω. The sharpest bound for almost certain stability is provided by Caughey and Gray. They use a Lyapunov-type approach to obtain sufficient conditions guaranteeing almost sure stability of Eq. (24). Infante and Plaut[13] used a Lyapunov technique to investigated the stability of a column subjected to a stationary random process, and concentrated their study to ergodic processes with a symmetric probability distribution. They obtain bounds for "almost pure" stability that are linearly proportional to the damping in the system. Since the results of Infante and Plaut are sharper, they will be considered as the representative bounds for almost sure stability for the comparison with our mean-square stability.

Figure 3 shows the behavior of the sufficient bounds for mean-square global stability, which implies almost sure stability. It is evident that the bounds for mean-square stability are not as extensive as those for "almost sure" stability. This reflects the inherent differences in the two respective types of convergence, coupled with the nature of the different

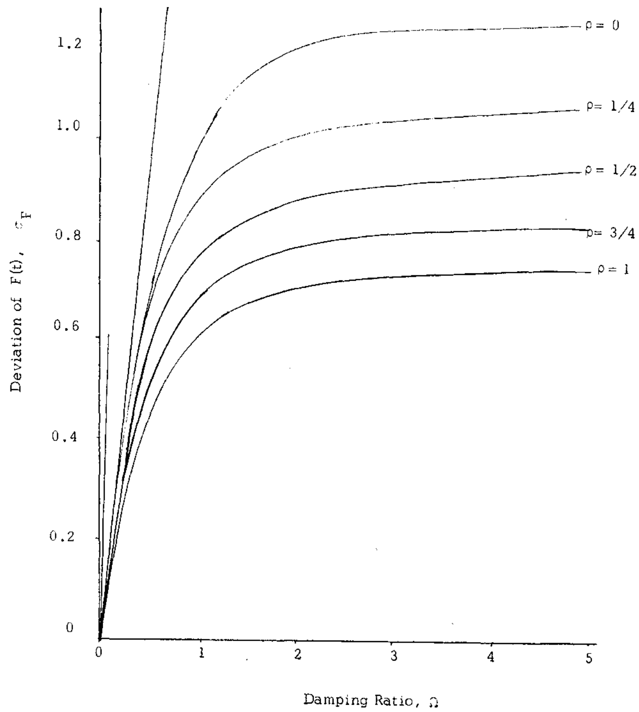

Fig. 3 Mean square stability bounds for gaussian processes.

Lyapunov Functions selected by the authors. Another important point to be made is that Infante and Plaut show that for "almost-sure" stability, $E[f(^2t)] \to \infty$ as $\Omega \to \infty$, which, from the physical standpoint is what one would expect. To the present time we have been unable to prove this assertion for mean-squared stability. However, the concept of mean-squared stability considered in this paper relates more statistical information of the excitation to stability bounds.

Symmetrical Probability Distributions

Many random processes obey probability distributions that are symmetrical about some mean value. We consider the case where we have an arbitrary symmetrically distributed stochastic excitation, which can be characterized by letting the process consist of a fixed mean value R_m and a random fluctuation $r(t)$ about this mean, i.e.,

$$R(t) = R_m + r(t) \quad (52)$$

$$R_m = E[R(t)] \quad (53)$$

Substituting this into Eq. (26), we obtain the following:

$$\frac{d}{d\xi}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ (2\beta R_m - 1) & -2\Omega \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + z\beta r(t)\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad (54)$$

$$\therefore \mathbf{z}' = [\tilde{A}^* + G(t)\tilde{B}]z \quad (55)$$

$$\therefore \mathbf{z} = \tilde{D}^*(t)\mathbf{z} \quad (56)$$

We can similarly obtain conditions on the original fluctuation $r(t)$

$$E[|r(t)|] + 0.895\rho\sigma_{|r|} < 2(R - Rm)/\{[(1 + 1/4\Omega^2)(2 - Rm/R^0)^2]^{1/2}\} \quad (57)$$

for $R_m < R^0$ where R^0 is the static buckling load.

In Fig. 4, the behavior of conditions Eq. (57) is investigated, where $G(t)$ represents the normalized fluctuation and

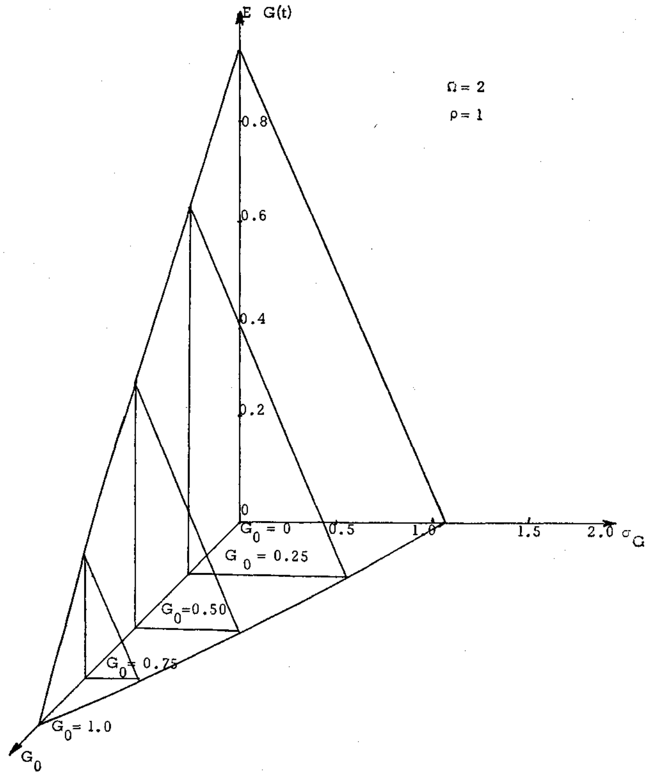

Fig. 4 Regions of stability for symmetric probability distribution.

G_0 the normalized mean value of the excitation. The parameters being varied are the correlation coefficient between the fluctuation measure $G(t)$ and the output measure n^2 along with the damping coefficient taking two different values.

The region enclosed by the surfaces developed and the coordinate planes are now our regions of mean square stability. It is apparent that as the mean value of the symmetric excitation G_0 increases, the corresponding fluctuation $G(t)$ must maintain more stringent statistical requirements in order to insure mean-square stability. As ρ decreases, the fluctuation can have a larger deviation and the dispersion about the mean value is increased. When Ω is decreased the general requirements for mean square stability become more severe, as expected.

As specific examples of symmetric processes, we shall consider two particular types of symmetric distributions. First we consider the gaussian process. Since $E[G(t)] = 0$ we

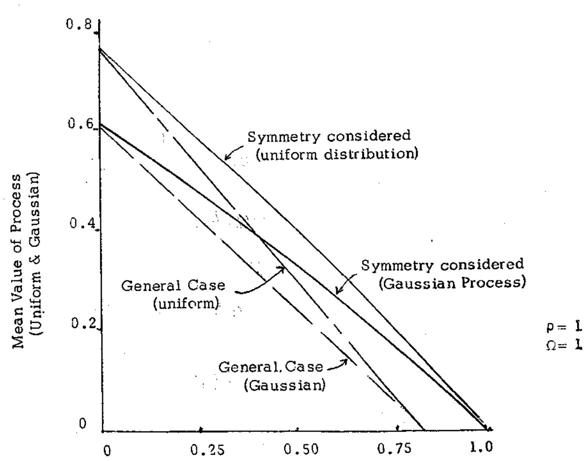

Fig. 5 Region of stability for gaussian and uniform probability distribution.

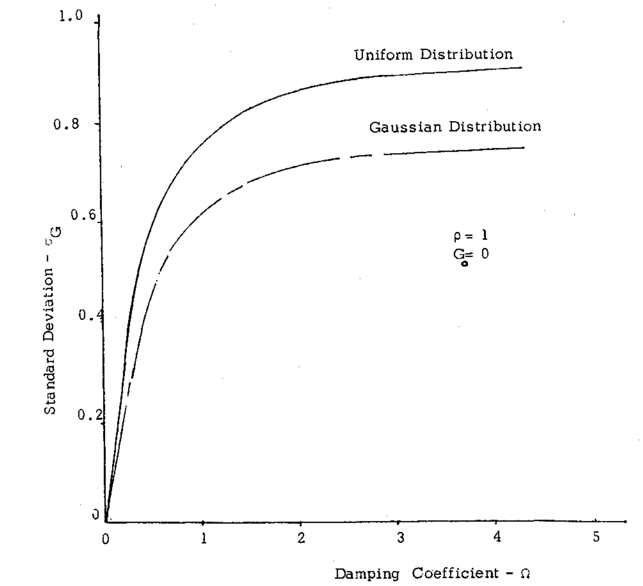

Fig. 6 Comparison between gaussian and uniform distributions.

can write Eq. (57) as

$$[0.798 + 0.540]\sigma G < \gamma(1 - G_0)/[1 + (1/4\Omega^2)(1 - G_0)^2] \quad (58)$$

for $G_0 < 1$.

These results are plotted Fig. 5 and regions of stability are developed. We consider only the case where the correlation coefficient is maximum $\rho = 1$ giving a minimum bound. It is understood that the regions of stability will increase in scope with a decrease in correlation between the input measure, therefore, this variation will not be taken into consideration in any of the subsequent plots. It is seen from this figure that there is a nonlinear relationship between the mean and variance of the gaussian excitation. As expected, the regions of stability decrease with a decrease in damping coefficient but the decrease is not as rapid as in the case when the excitation is treated as in the case when the excitation is treated as a general stochastic process.

The second example of symmetric processes we will consider is that the process has a uniform distribution. Since the uniform distribution can also be written in the form shown in Eq. (51) we have, in normalized form,

$$\tfrac{1}{4}[2(3)^{1/2} + 0.895\rho] < 2(1 - G_0)/[1 + \{1 + (2 - G_0)^2\}^{1/2}]$$

These results are plotted and regions of stability are developed in Fig. 5. In Fig. 7 where a comparison is made between the uniform distribution and the gaussian distribution, plotted for $G_0 = 0$. The degree to which the bound for the uniform case is larger than that for the gaussian case is readily seen. The apparent reason for this is that if, for a given fluctuation, we assume a gaussian probability law, due to the nature of the law, there will exist a small but finite probability that the fluctuation will assume a larger value. This is because the guassian distribution has a range from $-\infty$ to $+\infty$ and it possesses extensive "tails." The uniform distribution has no "tails" at all, hence there is no finite probability that large fluctuations will exist.

In conclusion, it should be noted that the preceding approach determined sufficient bounds on the statistics of the excitation for incipient instability. That is, an increase in the statistical characteristics of the excitation beyond the critical values presented previously does not mean the plate is physically unable to resist radial loads. This means that our assumptions of small deflections, and consequent governing differential equation, is no longer valid. We must investigate the nonlinear characteristics of the system and this becomes a natural extension of the method used herein.

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Dynamic Stability of Cylindrical Propellant Tanks

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Dynamic instability and associated parametric resonance is a dominant form of response in a longitudinally excited cylindrical shell containing liquid. The present paper is devoted to a theoretical and experimental study of its occurrence in a cylindrical system which includes the influences of axial preload, ullage pressure, partial liquid depth, and a finite top impedance. Donnell shell theory and a modified Galerkin procedure are utilized to formulate equations which govern the stability of perturbations superimposed on an axisymmetric initial state of response. Stability boundaries are computed for a range of parameters affecting the region of principal parametric resonance and are compared with experimental results. It is found that liquid depth, top impedance, and ullage pressure have a strong influence on stability, while the effects of axial preload are relatively insignificant.

Nomenclature

a	= radius of the shell
c_0	= speed of sound in the liquid
c_s	= E/ρ_s, speed of stress waves in the shell
E	= modulus of elasticity
g	= standard acceleration of gravity
H	= h/a, nondimensional liquid depth
H_s	= h_s/a, nondimensional thickness of shell
I_z	= mass moment of inertia of top weight about z axis
l	= length of the shell
m	= one-half of the number of circumferential nodes; $\cos(m\theta)$
$N_{xxd}^*, N_{\theta\theta d}^*$	= dynamic part of initial-state stress resultants [nondimensionalized by $(l - \nu^2)/Eh_s$]
$N_{xxs}^*, N_{\theta\theta s}^*, N_{x\theta s}^*$	= static part of initial-state stress resultants [nondimensionalized by $(l - \nu^2)/Eh_s$]
n	= axial wave number; $\sin n\pi x/l$
P_r	= nondimensional pressure loading on shell, p_r/E
P_0, p_0	= axial preload, ullage pressure
R, θ, X	= cylindrical coordinates (space-fixed) nondimensionalized by a
U, V, W	= shell displacements u, v, w, nondimensionalized by a
X_0	= nondimensional amplitude of axial excitation ($X_0 = \hat{x}_0/a$)
Z_0	= top acceleration impedance (force/acceleration)
β	= density parameter $\rho_l a/\rho_s h_s$
ν	= Poisions ratio
Φ	= velocity potential, nondimensionalized by $\omega_0^2 a^2/\omega_r$
ρ_l, ρ_s	= mass densities of liquid and shell
τ	= nondimensional time, $\tau = \omega_r t$
ω_0^2	= liquid parameter c_0^2/a^2
ω_r, ω	= response and excitation frequencies
ω_k	= natural frequency of m, k-th mode
Ω_i^2	= designated frequency, nondimensionalized by a^2/c_s^2
$\tilde{\Omega}_i^2, \bar{\Omega}_i^2$	= designated frequencies, nondimensionalized by $(1 - \nu^2)a^2/c_s^2$ and by a^2/c_0^2, respectively

Superscripts

$(\hat{\ })$	= the amplitude of ( )
$(\dot{\ })$	= $(d/d\tau)$ ( ), $\tau = \omega t$
$(\ )^p$	= related to initial-state response

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